ON THE CONSTRUCTION OF COMPLETE SETS OF F-SQUARES OF ORDER n = 2k, n = 2^sk, p^s, AND n = 4t WITH LATIN SQUARES

By

Walter T. Federer, Departments of Biometrics and Statistical Sciences, Cornell University

ABSTRACT

Several methods of constructing combinatorially orthogonal F-squares are discussed. Some limitations of results in the literature are illustrated. Sum-of-squares orthogonality is discussed and illustrated with numerical examples. SAS computer codes and outputs for the examples are included. Areas of future research and some open questions are presented. In particular, questions about the generality of results given by Anderson, Federer, and Seiden (1974) are raised. The NOA module of the Gendex toolkit was used to add one orthogonal F-square with two symbols, F(12, 6), to the set of five orthogonal Latin squares of order 12, OLS(12, 5) + the six F(12, 6) squares given by Mandeli (1978). A new property described by Federer (2000a, 2000b) is sum-of-squares orthogonality. This property is discussed and illustrated with examples where it is shown how to obtain complete sets of sum-of-squares orthogonal F-squares.

Key words: Combinatorial orthogonality, Sum-of-squares orthogonality, Balanced incomplete block design, Permutations, Difference method, Gendex, NOA module.

BU-1597-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca, New York 14853, May 2002.

INTRODUCTION

It is well-known that complete sets of orthogonal F-squares exist for n = p^s for p = a prime number. The associated projective geometry is also well-known (See references under the Literature section.). The question arises as to what can be done when n, the order of the F-square, is not a power of a prime number. The type of orthogonality considered here is combinatorial. We show that if another type of orthogonality is considered complete sets of F-squares may be obtained for non-prime power numbers. Federer (2000b) has defined an orthogonality property called sum-of-squares orthogonality. Using this property he has shown how to construct complete sets of F-squares for products of prime numbers.

Anderson *et al.* (1974) present a number of theorems and four different methods for constructing orthogonal F-squares, *i.e.*, permutation method, balanced incomplete block design method, difference method, and orthogonal array method. A question is raised regarding the generality of their results. In particular, are their methods and/or theorems only for n = 6 and not for 2k?

An F-square of order n is a square with n rows and n columns. An F-square with two symbols, say, is denoted as F(n, n/2) where a symbol occurs n/2 times in each row and in each column. An F-square with p symbols is written as F(n, n/p) with each symbol occurring n/p times in each row and in each column. A modified F-square as defined by Federer (2000a, 2000b) is an $n \times n$ square with p symbols where each symbol occurs equally frequent in a row or in a column but not both. The p symbols occur equally frequent in the square.

COMPLETE SETS OF F-SQUARES FOR $n = 2^s$ WITH A CYCLIC LATIN SQUARE

Schwager *et al.* (1984) have shown how to embed a cyclic Latin square of order $n = 2^s$ into a complete set of F-squares with 2 symbols, where an F-square with 2 symbols is denoted as F(n, n/2). It is known that a cyclic Latin square of even order has no combinatorial orthogonal mate. However, a cyclic Latin square of order $n = 2^s$ may be embedded in a complete set of orthogonal F(n, n/2) squares. An open question is how to combine the F(n, n/2) squares to form F(n, n/4), F(n, n/8), etc. squares. When this result is available, several different sets of orthogonal arrays (codes) are possible.

COMPLETE SETS OF F-SQUARES FOR $n = p^s$, p A PRIME NUMBER, WITH p - 1 CYCLIC LATIN SQUARES

Federer (1992) presents a method for embedding p - 1 cyclic Latin squares of order $n = p^s$, p a prime number, in a complete set of $F(n, n/p^{s-1})$ squares. For example for n = 9, there are p - 1 = 2 cyclic orthogonal Latin squares. These together with 24 F(9, 3) squares form a complete set of two orthogonal Latin squares and 24 F-squares with three symbols. Although it is shown how to do this for any particular n, a general procedure needs to be supplied in the form of theorems and/or algorithms. Li (1994) has presented a method for constructing a set cyclic Latin squares and orthogonal F-squares when $n = p^2$.

COMPLETE SETS OF F-SQUARES FOR n = 4t

If a Hadamard matrix exists, then $(4t - 1)^2$ F(4t, 2t) squares may be constructed to make a complete set of orthogonal F-squares. The row single degree of freedom contrasts may be those obtained from the Hadamard matrix of -1s and +1s, where 0 is replaced by -1 to make it a contrast. Likewise the column single degree of freedom contrasts may be obtained from a Hadamard matrix. The products of the row and column contrasts provide the $(4t - 1)^2$ interaction single degree of freedom interaction contrasts. These single degree of freedom interaction terms may be used to construct F(4t, 2t) squares. Since the products of two Hadamards is also a Hadamard, the resulting $(4t - 1)^2$ F(4t, 2t) squares are combinatorially orthogonal (Federer, 1977).

Also, if there is an orthogonal set of r Latin squares of order n = 4t, is it possible to complete the set with F(4t, 2t) squares? Mandeli (1978) added six F(12, 6) squares to the set of five orthogonal Latin squares for n = 12. To determine if there are more orthogonal F(12, 6)-squares that can be added to the set, it is suggested that one method of construction is to use the NOA module of the toolkit Gendex with the module set to accommodate 256 runs. For n = 12, a complete set of five orthogonal Latin squares of order 12, OLS(12. 5), and 66 F(12, 6) squares could be obtained to make a complete set. The row by column interaction for n = 12 has 121 degrees of freedom. Allocating 5(11) = 55 for the five Latin squares leaves 121 - 55 = 66 degrees of freedom for the 66 F(12, 6) squares.

In an attempt to construct a number of F(12, 6) squares orthogonal to the OLS(12, 5) set, the NOA module of the Gendex toolkit was utilized. The input array or matrix was the 144 × 7 array obtained from the OLS(12, 5) set of Latin squares. One additional 2-level factor was added to the array by using the random seed -313153417. This array was used as the input array was the output from this step to form a 144 × 8 array. One 2-level factor which was orthogonal to the input array was added using the random seed -313153417. Using this 144 × 9 array as the input array, another 2-level factor orthogonal to this array was found using the random seed -1847435164. Using this 144 × 10 array as the input array, one 2-level factor almost orthogonal to the array was found using any of the random seeds -313153417, 1442316452, or -1061417177. This -2 in the X'X matrix needs to be zero in order for this fourth F(12, 6) square to be orthogonal to the input array. The output from this calculation is given in Table 1. It is not known if other random seeds will produce additional F(12, 6) squares orthogonal to the OLS(12, 5) set plus the three F(12, 6) squares obtained. As can be observed from the X'X matrix, one -2 occurs in the last column. It appears that the stopping point in the NOA module needs to be raised if additional 2-level factors or even adding more than one 2-level factor at a time can be done.

Using the OLS(12, 5) set plus the six F(12, 6) squares found by Mandeli (1978) as the input array, the NOA module of Gendex was used in an attempt to find an additional F(12, 6) square orthogonal to this array. The random seed 1442316452 was used. The output obtained at this stage of using the NOA module of the Gendex toolkit is given in Table 1A. As may be observed from the X'X matrix and the correlation matrix, the resulting F(12, 6) square is nearly orthogonal. It is not known if the Mandeli (1978) set is locked or not. However it would appear that more orthogonal F(12, 6) squares are possible.

SETS OF F-SQUARES FOR n = 2k

Anderson *et al.* (1974) present four methods for constructing F(2k, 2) squares. They are the method of permutations, balanced incomplete block design method, the difference composition method, and the method of orthogonal arrays. The method of permutations produced the largest number of mutually pairwise orthogonal F(2k, k) squares for 2k = 6. We discuss this method herein and pose several unanswered questions about the various methods. They present the following Theorem 2.1:

Theorem 2.1: *Let n = 2t + 1. A set of r permutations of integers -t through t produces r - 1 orthogonal F(n+1, 2) squares, if when placed in an r × m array,*

- *(a) differences with the first row mod(n + 1) reproduce -t through t, and*
- *(b) differences of any other pair mod(t + 1) produce 0 one time and 1, 2, ..., t each two times.*

It is not clear what they mean by first row but presumably they mean the ordered sequence -t through t. Also, they mention mod(6) for permutations but this doesn't make sense as only three symbols are required for the resulting F-squares. We shall try to ascertain what they mean by studying the permutations given. Note they merely present a set of permutations with no method of obtaining a set of permutations for other n, say 10, or even any indication that they exist.

For n = 2t + 1, r permutations of the numbers -t through t are required. The various permutations must satisfy two conditions. According to condition (a) of the Theorem, the differences with the ordered permutation -t through t with r - 1 (our guess) of the other permutations must reproduce the integers -t through t. Condition (b), we think, is that differences of the first r - 1 permutations with the rth permutation modulo(t + 1) must reproduce the integers 0 once and 1, 2, ..., t each two times. To illustrate, let n = 2t + 1 = 5. Then, r = 5 permutations of the integers -2, -1, 0 , 1, and 2, given by Anderson *et al.* (1974) are:

Permutation					
0:	-2	-1	0	1	2
1:	0	-2	1	-1	2
2:	-1	-2	2	1	0
3:	-2	0	2	-1	1
4:	2	-1	1	0	-2

The pairwise differences of the above n = 2t + 1 = 5 permutations are:

0 - 1:	-2	1	-1	2	0							
0 - 2:	-1	1	-2	0	2							
0 - 3:	0	-1	-2	2	1							
1 - 2:	1	0	-1	-2	2							
1 - 3:	2	-2	-1	0	1							
2 - 3:	1	-2	0	2	-1							
0 - 4:	-4	0	-1	1	4	=	2	0	2	1	1	modulo(3)
1 - 4:	-2	-1	0	-1	4	=	1	2	0	2	1	modulo(3)
2 - 4:	-3	-1	1	1	2	=	0	2	1	1	2	modulo(3)
3 - 4:	-4	1	1	-1	-3	=	2	1	1	2	0	modulo(3)

Thus, condition (a) and condition (b) *appear to be* satisfied by the above set of permutations. It is stated that "any pair of the first two rows mod(6) reproduce {-2, -1, 0, 1, 2}". What is meant by this statement? There was no need for mod(6) since the differences for the first six sets above reproduce the original sequence of numbers. Also, the first set of six differences mod(3) produce 0 once and 1 and 2 each two times as for condition (b). Does this mean that condition (a) could be omitted? The last set of four differences are the negative values of those given by Anderson *et al.* (1974) except for a typo for the 2 - 4 set of differences where they should have a -2. Here they say to use mod(6) for the 0 - 4 differences but since only symbols 0, 1, and 2 are wanted for the F-square, mod(3) should be used for this set. However, if mod(6) is used first and then mod(3), the result is the same. Why use mod(6) when it is not needed for this sequence? Is mod(n+1) required for larger values? Their proof throws no light on these questions. It appears that a clarified form of these statements should appear in the Theorem, and the proof should give some indication of why such a set of permutations works. The proof of the Theorem given by Anderson *et al.* (1974) relates to constructing the set of r - 1 orthogonal F-squares but nothing about constructing the set of r permutations of -t through t. In order for their result to be usable, a method of constructing the set of r permutations is required.

Note that the above 5 × 5 array of permutations are not in a Latin square arrangement. For n = 9, attempting to find a 9 × 9 array of such permutations using a Latin square arrangement was unsuccessful. A method for obtaining permutations of -t through t that satisfy the two conditions is required.

Once such an r × m array is obtained, F(2k, 2) squares may be formed as follows. Denote the r × m array as the matrix M. Append a first row and a last column of zeros to the matrix M to form a (r + 1) × (m + 1) matrix. Add this matrix to an h × J, J is matrix of ones of (r + 1) rows and (m + 1) columns. Let h = 0, 1, 2, …, 2t + 1. The first row of the resulting square is the row number of the F-square, the second row denotes the column number, and each of the remaining rows modulo(t + 1) produces an F(2k, 2) square.

For the number 10, the following set was obtained but only single pairs of the permutations satisfy condition (a):

-4	-3	-2	-1	0	1	2	3	4
-3	0	-4	1	4	-2	2	-1	3
-2	-4	1	3	0	-3	-1	4	2
-1	-2	2	-3	-4	0	4	3	1
0	-1	1	-4	-2	-3	3	2	4
-4	-2	-3	3	2	4	-1	1	0
-3	-4	1	3	0	-2	4	-1	2
-2	1	-3	-4	3	-1	2	4	0
-1	1	0	-4	-2	-3	2	4	3
0	-1	1	-3	-4	-2	2	4	3
-3	-1	1	-4	4	0	-2	3	2
-4	1	-1	2	-2	-3	4	0	3
0	-3	1	-4	2	-1	-2	4	3
-2	0	-1	-4	4	-3	2	1	3

The 13 permutations below the first ordered permutation are each permutations whose differences reproduce the numbers -4 through 4 with the ordered permutation but none of them have this property with each other. That is, a set larger than two of permutations satisfying condition (a) was not found. Note that for n = 10, a set of r = 9 permutations should be possible. This would allow construction of r - 1 = 8 pairwise orthogonal F(10, 2) squares. For n = 14, there should be 13 permutations of the kind envisaged by the Theorem. For n = 18, there should be 17 such permutations. But the question is, how are they constructed? Is this method unique for n = 6?

The four F(6, 2) squares produced using the above set of permutations and method of construction are:

1 2 2 0 0 1	2 1 0 0 2 1	2 2 1 0 1 0	1 0 1 0 2 2
2 2 0 0 1 1	2 0 2 1 1 0	1 0 0 2 1 2	0 2 1 2 1 0
2 0 0 1 1 2	1 0 1 0 2 2	0 2 1 1 0 2	1 1 0 2 0 2
0 0 1 1 2 2	0 2 1 2 1 0	0 1 0 2 2 1	0 2 2 1 0 1
0 1 1 2 2 0	1 1 0 2 0 2	2 1 2 1 0 0	2 1 0 0 2 1
1 1 2 2 0 0	0 2 2 1 0 1	1 0 2 0 2 1	2 0 2 1 1 0

Interchanging the role of rows and columns leads to another set of F(2k, 2) squares which are pairwise orthogonal to the above set. Thus, for the number six, eight F(6, 2) squares are constructed. Anderson *et al.* (1974) prove that this set is locked and cannot be extended further. As we shall show in a later section, three modified F(6, 2) sum-of-squares orthogonal squares can be constructed to complete the set.

Since the desired permutations of the sequence -4 through 4 was not obtained, it was decided to look at another construction method of Anderson *et al.* (1974), *i.e.,* the balanced incomplete block design. The F(10, 2)-square comparable to their (3.1) is:

1	0	0	1	1	2	2	3	3	4	4
2	0	0	1	1	2	2	3	3	4	4
3	1	1	2	2	3	3	4	4	0	0
10	1	1	2	2	3	3	4	4	0	0
4	2	2	3	3	4	4	0	0	1	1
8	2	2	3	3	4	4	0	0	1	1
5	3	3	4	4	0	0	1	1	2	2
9	3	3	4	4	0	0	1	1	2	2
6	4	4	0	0	1	1	2	2	3	3
7	4	4	0	0	1	1	2	2	3	3

Using the BIB module of Gendex to obtain a design comparable to their (3.2), the balanced incomplete block for v = 10, k = 2, and r = 9 for the above is

Replicate I	1	3	4	5	6
	2	10	8	9	7
Replicate II	1	2	3	4	7
	8	9	6	5	10
Replicate III	1	2	3	6	7
	10	4	5	8	9
Replicate IV	1	2	3	4	8
	6	5	7	9	10
Replicate V	1	2	3	5	6
	4	8	9	7	10
Replicate VI	1	2	4	5	6
	7	3	10	8	9
Replicate VII	1	2	3	4	5
	9	10	8	7	6
Replicate VIII	1	2	3	7	9
	5	6	4	8	10
Replicate IX	1	2	4	5	8
	3	7	6	10	9

The following first columns, comparable to their (3.3), are obtained from each of the replicates as:

I	II	III	IV	V	VI	VII	VIII	IX
0	0	0	0	0	0	0	0	0
0	2	1	4	2	4	3	3	1
1	0	0	0	0	0	0	0	0
1	3	2	3	2	1	1	4	4
2	1	1	1	1	2	1	1	2
2	4	3	4	3	1	2	2	4
3	2	4	2	3	3	2	4	3
3	3	2	3	4	2	4	2	1
4	4	4	2	4	4	3	3	2
4	1	3	1	1	3	4	1	3

The third and sixth rows above look peculiar. The incidence relationships of any two columns is a balanced arrangement for n = 6. The incidence arrangement of two columns, say I and II, is

		Replicate II				
		0	1	2	3	4
Replicate I	0	1		1		
	1	1			1	
	2		1			1
	3			1	1	
	4		1			1

Is this considered to be a balanced arrangement in the sense of Anderson *et al.* (1974)? Using the first columns above to construct F(10, 2)-squares did not result in orthogonal F-squares. Does this mean that the balanced incomplete block design method of Anderson *et al.* (1974) is unique for the number 6, and not applicable to other n?

COMPLETE SETS OF SUM-OF-SQUARES ORTHOGONAL F-SQUARES

Federer (2000a, 2000b) defined a concept called sum-of-squares orthogonal (SSO) and demonstrates how to construct a complete set of SSO F-squares for 2^np, p a prime number. He presents a second method called single-degree-of-freedom orthogonality from which a complete set of sum-of-squares orthogonal F-squares may be obtained. SSO is defined as:

When the sum of squares for any main effect or interaction from a complete factorial is the same as that for the corresponding F-square or modified F-squares, this property is denoted as sum-of-squares orthogonality, SSO.

A data set for six treatments with six replicates designed as a randomized complete block experiment design is used to demonstrate the procedure. The replicates are called rows and the treatments columns to form a 6 × 6 square. The rows and columns are each considered to be a 2 × 3 factorial of factors A and B for rows and C and D for columns. The row by column interaction may be partitioned into the interactions of factors A, B, C, and D. The data and SAS/GLM code are given in Table 2 and the analysis of variance (ANOVA) obtained is given in Table 3. The method of constructing the F-squares and modified F-squares is (Federer, 2000a, 2000b):

Given an n × n square, where n is a product of prime numbers, partition n into its multiples. Then obtain an ANOVA partitioning of degrees of freedom for the factorial formed by the prime numbers. For each main effect and interaction form the corresponding F-squares and modified F-squares as for prime numbers except that modulo the largest number is used.

The 19 F-squares and modified F-squares are given in Table 2. For example, the AB, the BC, ABC, ABCD, and ABCD2 squares are, respectively:

000000	000111	000111	012120	021102
111111	111222	111222	120201	102210
222222	222000	222000	201012	210021
111111	000111	111222	120201	102210
222222	111222	222000	201012	210021
000000	222000	000111	012120	021102

The symbols 0, 1, and 2 appear equally frequent in the squares. They appear equally frequent in a row or in column but not both. This is why AB, BC, and ABC are called modified F-squares. ABCD and ABCD2 are regular F-squares. Note that the F-squares formed by levels of A, B, C, D, AC, and others are regular F-squares.

In Table 3 it may be noted that these F-squares and modified F-squares account for all of the sums of squares. The Type I sum of squares SAS/GLM is used. Type III or IV sums of squares are not appropriate as they take combinatorial orthogonality into account. This set of 19 F-squares is a complete set of sum-of-squares orthogonal F-squares.

COMPLETING A SET OF COMBINATORIALLY ORTHOGONAL F-SQUARES WITH SUM-OF-SQUARES ORTHGONAL F-SQUARES

Federer (1975) (Also, see Finney, 1982) presented the following set of seven F(6, 2) squares and one Latin square which are combinatorially orthogonal. He used the eight F(6, 2) squares of Anderson *et al.* (1974) to construct the set.

	Columns					
Row	1	2	3	4	5	6
1	2011	1102	1221	0000	0212	2120
	0110	1110	2221	3221	4002	5002
2	2222	0110	1012	2101	0021	1200
	1021	0021	3102	2102	5210	4210
3	0110	2222	2101	1012	1200	0021
	2200	3200	4011	5011	0122	1122
4	0201	1020	2210	0122	1111	2002
	3012	2012	5120	4120	1201	0201
5	1020	0201	0122	2210	2002	1111

	4101	5101	0212	1212	2020	3020
6	1102	2011	0000	1221	2120	0212
	5222	4222	1000	0000	3111	2111

The Latin square of order six (first number in the second set of four numbers) was obtained as the interaction of an F(6, 3) and one of the original eight F(6, 2) squares. Using the same data set as before, the sums of squares for each of the seven F-squares and the Latin square are given in Table 4. There are six degrees of freedom remaining as Error with a sum of squares of 1162.3333.

Letting rows be a two-level by three-level factorial of Factors A and B, respectively and likewise for columns with factors C and D, modified F-squares are constructed. These are constructed as the interactions of AF1 = A × F1, AF2 = A × F2, and CF3 = C × F3, where F1, F2, and F3 are the first three F-squares formed from the array above and as described in Table 4. The data set, SAS/GLM code, and output are given in Table 4. It may be noted that two of these interactions may be with rows or with columns but the other one must be with columns or rows in order to account for the remaining sums of squares. These three modified F-squares account for all of the "error" sum of squares to complete the set of F-squares. Note the Type I and Type III sums of squares are identical for the seven F(6, 2) squares and the Latin square (treat) as these form a combinatorially orthogonal set. This is not true when the three modified F-squares are added since combinatorial non-orthogonality has been introduced.

SOME LITERATURE

Anderson, D. A., W. T. Federer, and E. Seiden (1974). On the construction of F(2k, 2) squares. BU-500-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca, New York 14853.

Cheng, C. S. (1980). Orthogonal arrays with variable numbers of symbols. Annals of Statistics 8:447-453.

Federer, W. T. (1972). Construction of classes of experimental designs using transversals in Latin squares and Hedayat's sum composition method. In *Statistical Papers in Honor of George W. Snedecor* (Editor: T. A. Bancroft), Iowa State University Press, Ames, Iowa, pp. 91-114.

Federer, W. T. (1975). On the construction of F-squares and single degree of freedom contrasts. BU-566-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca, New York 14853.

Federer, W. T. (1977). On the existence and construction of a complete set of orthogonal F(4t, 2t, 2t)-squares design. Annals of Statistics 5:561-564.

Federer, W. T. (1992). F-square geometries for $s = p^n$ illustrated with s = 4, 8, and 9. BU-1173-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca, New York 14853.

Federer, W. T. (2000a). Complete sets of F-squares for products of prime numbers. BU-1500-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca, New York 14853.

Federer, W. T. (2000b). Addendum to BU-1500-M: Complete sets of F-squares. BU-1536-M in the Technical Report Series of the Department of Biometrics, Cornell University, Ithaca, New York 124853.

Federer, W. T., A. Hedayat, and J. P. Mandeli (1984). Pairwise orthogonal F-rectangle designs. Journal of Statistical Planning and Inference 10:365-374.

Federer, W. T., A. Hedayat, E. T. Parker, B. L. Raktoe, E. Seiden, and R. J. Turyn. (1971). Some techniques for constructing mutually orthogonal Latin squares., Technical summary Report No. 1030, Mathematics research Center, University of Wisconsin.

Federer, W. T. and J. P. Mandeli (1986). Orthogonal F-rectangles, orthogonal arrays, and codes. Journal of Combinatorial theory, Series A, 43:149-164.

Finney, D. J. (1982). Some enumerations for the 6 × 6 Latin square. Utilitas Mathematica 21A:137-153.

Hedayat, A. (1969). On the theory, existence, non-existence, and the construction of mutually orthogonal F-squares and Latin squares, Ph. D. thesis, Cornell University.

Hedayat, A. S. and W. T. Federer (1970). An easy method of constructing partially replicated Latin square designs of order n for all n > 2. Biometrics 26:327-330.

Hedayat, A. S. and W. T. Federer (1970).On the equivalence of Mann's group automorphism method of constructing an O(n, n - 1) set and Raktoe's collineation method of constructing a balanced set of *l*-restrictional prime-powered lattice designs. Annals of mathematical Statistics 41:1530-1540.

Hedayat, A. S. and W. T. Federer (1984). Orthogonal F-rectangles for all even v. Calcuta Statistical Association Bulletin 33:85-92.

Hedayat, A., D. Raghavarao, and E. Seiden (1975). Further contributions to the theory of F-squares design. Annals of Statistics 3:712-716.

Hedayat, A. and E. Seiden (1970). F-squares and orthogonal F-square design: A generalization of Latin square and orthogonal Latin square design. Annals of Mathematics and Statistics 41:2035-2044.

Hedayat, A. and E. Seiden (1970). F-square and orthogonal F-square design: A generalization of Latin square and orthogonal Latin square design. Proc. Second Chapel Hill Conference on Combinatorial Mathematics and its Applications. Pp. 261-275.

Hedayat, A., E. Seiden, and W. T. Federer (1972). Some families of multistage experiments: Mutually balanced Youden designs when the number of treatments is prime or twin primes. Institute of Annals of Mathematical Statistics 43:1517-1527.

Gendex (2001). http//designcomputing.hypermart.net/gendex.

Li, Yanfen (1994). Construction of a complete set of F-squares of order p^2. M. S. Thesis, Cornell University.

Mandeli, J. P. (1975). Complete sets of orthogonal F-squares. M. S. thesis, Cornell University.

Mandeli, J. P. (1978). Contributions to the theory of F-square and F-cube designs. Ph. D. Thesis, Cornell University.

Mandeli, J. P. and W. T. Federer (1983). An extension of MacNeish's theorem on the construction of sets of pairwise orthogonal F-squares of composite order. Utilitas Mathematica 24:87-96.

Mandeli, J. P. and W. T. Federer (1984). On the construction of mutually orthogonal F-hyperrectangles. Utilitas Mathematica 25:315-324.

Mandeli, J. P. and W. T. Federer (1984). Complete sets of orthogonal F-squares of prime power order with differing numbers of symbols. In *Experimental Design, Statistical Models, and Genetic Statistics*, (Editor: Klaus Hincelmann), Marcel Dekker, Inc., New York and Basel, pp. 45-58.

Mandeli, J. P., F. C. Lee, and W. T. Federer (1981). On the construction of orthogonal F-squares of order n from an orthogonal array (n, k, s, 2) and an)LS(n, t) set. Journal of Statistical Planning and Inference 5:267-272.

MacNeish, (1922). Euler's squares. Annals of Mathematics 23:221-227.

Raktoe, B. L. and W. T. Federer (1985). Lattice square approach to construction of mutually orthogonal F-squares. Annals of the Institute of Statistical Mathematics 37:329-336.

Schwager, S. J., W. T. Federer, and J. P. Mandeli (1984). Embedding cyclic Latin squares of order 2^n in a complete set of orthogonal F-squares. J. of Statistical Planning and Inference 10:207-218.

Schwager, S. J., W. T. Federer, and B. L. Raktoe (1984). Nonisomorphic complete sets of orthogonal F-squares and Hadamard matrices. Communications In Statistics, Theory and Methods 13:1391-1406.

Table 1. Near orthogonal array for five orthogonal Latin squares of order 12 and four F(12, 6) squares.

```
NOA 2.0: Construct orthogonal and near-orthogonal arrays with mixed levels
(C) 2001 Design Computing (URL: http://designcomputing.hypermart.net/gendex)

Note: design for m=11 and n=144.

try #               1
seed                1442316452
# of iterations     8
ave                 7244899.2727
max of s(i,j)       4356
freq of max         21
D-eff               0.5841
A-eff               0.4327

Factor levels:
```

0	0	0	0	0	0	0	-1	1	1	-1
0	1	1	1	1	1	1	-1	1	-1	1
0	2	2	2	2	2	2	1	-1	1	-1
0	3	3	3	3	3	3	-1	1	-1	-1
0	4	4	4	4	4	4	1	-1	1	-1
0	5	5	5	5	5	5	-1	-1	1	1
0	6	6	6	6	6	6	1	1	-1	-1
0	7	7	7	7	7	7	-1	1	1	1
0	8	8	8	8	8	8	1	-1	-1	1
0	9	9	9	9	9	9	-1	1	-1	1
0	10	10	10	10	10	10	-1	-1	-1	-1
0	11	11	11	11	11	11	-1	-1	1	-1
1	0	5	6	3	10	2	1	1	-1	1
1	1	0	7	4	11	3	-1	-1	1	-1
1	2	1	8	5	6	4	1	1	1	1
1	3	2	9	0	7	5	1	1	-1	1
1	4	3	10	1	8	0	-1	-1	1	1
1	5	4	11	2	9	1	1	1	-1	-1
1	6	11	0	9	4	8	-1	1	1	-1
1	7	6	1	10	5	9	1	-1	-1	1
1	8	7	2	11	0	10	-1	1	1	-1
1	9	8	3	6	1	11	1	1	1	-1
1	10	9	4	7	2	6	1	1	1	1
1	11	10	5	8	3	7	1	-1	1	1
2	0	4	10	6	5	7	-1	-1	-1	-1
2	1	5	11	7	0	8	1	1	1	1
2	2	0	6	8	1	9	1	1	-1	-1
2	3	1	7	9	2	10	1	1	1	1
2	4	2	8	10	3	11	1	1	-1	-1
2	5	3	9	11	4	6	1	-1	-1	1
2	6	10	4	0	11	1	1	1	1	-1
2	7	11	5	1	6	2	-1	-1	1	-1
2	8	6	0	2	7	3	-1	1	1	-1
2	9	7	1	3	8	4	-1	-1	-1	1
2	10	8	2	4	9	5	1	-1	-1	1
2	11	9	3	5	10	0	-1	1	1	-1
3	0	3	4	5	7	8	1	1	-1	1
3	1	4	5	0	8	9	-1	-1	-1	1
3	2	5	0	1	9	10	-1	1	1	1
3	3	0	1	2	10	11	-1	-1	-1	1
3	4	1	2	3	11	6	1	-1	1	-1
3	5	2	3	4	6	7	1	1	1	-1
3	6	9	10	11	1	2	-1	-1	-1	-1

3	7	10	11	6	2	3	-1	-1	-1	1
3	8	11	6	7	3	4	-1	1	-1	1
3	9	6	7	8	4	5	1	1	1	1
3	10	7	8	9	5	0	1	-1	-1	1
3	11	8	9	10	0	1	1	-1	-1	1
4	0	2	11	9	1	4	-1	1	1	-1
4	1	3	6	10	2	5	-1	-1	1	1
4	2	4	7	11	3	0	-1	-1	-1	-1
4	3	5	8	6	4	1	-1	1	-1	1
4	4	0	9	7	5	2	-1	-1	-1	1
4	5	1	10	8	0	3	1	-1	1	-1
4	6	8	5	3	7	10	-1	-1	-1	-1
4	7	9	0	4	8	11	1	-1	1	1
4	8	10	1	5	9	6	-1	-1	-1	-1
4	9	11	2	0	10	7	1	1	-1	-1
4	10	6	3	1	11	8	1	-1	-1	-1
4	11	7	4	2	6	9	1	1	1	-1
5	0	1	5	7	9	11	1	-1	1	1
5	1	2	0	8	10	6	-1	-1	1	-1
5	2	3	1	9	11	7	1	1	-1	1
5	3	4	2	10	6	8	1	-1	-1	-1
5	4	5	3	11	7	9	-1	-1	1	1
5	5	0	4	6	8	10	-1	1	-1	-1
5	6	7	11	1	3	5	1	1	-1	1
5	7	8	6	2	4	0	-1	1	-1	-1
5	8	9	7	3	5	1	1	-1	-1	-1
5	9	10	8	4	0	2	1	-1	1	-1
5	10	11	9	5	1	3	-1	-1	1	1
5	11	6	10	0	2	4	1	1	-1	1
6	0	6	9	4	3	10	-1	-1	1	-1
6	1	7	10	5	4	11	-1	1	-1	-1
6	2	8	11	0	5	6	-1	-1	-1	-1
6	3	9	6	1	0	7	-1	-1	1	1
6	4	10	7	2	1	8	-1	1	1	-1
6	5	11	8	3	2	9	-1	1	1	1
6	6	0	3	10	9	4	1	-1	-1	1
6	7	1	4	11	10	5	-1	-1	1	1
6	8	2	5	6	11	0	-1	1	1	-1
6	9	3	0	7	6	1	-1	1	1	1
6	10	4	1	8	7	2	1	1	-1	-1
6	11	5	2	9	8	3	-1	1	-1	1
7	0	11	7	10	8	6	-1	1	-1	1
7	1	6	8	11	9	7	1	1	1	1
7	2	7	9	6	10	8	1	-1	1	-1
7	3	8	10	7	11	9	1	1	-1	-1
7	4	9	11	8	6	10	1	-1	1	-1
7	5	10	6	9	7	11	1	-1	1	1
7	6	5	1	4	2	0	1	-1	-1	-1
7	7	0	2	5	3	1	-1	-1	-1	1
7	8	1	3	0	4	2	1	-1	1	-1
7	9	2	4	1	5	3	1	-1	-1	1
7	10	3	5	2	0	4	-1	1	-1	1
7	11	4	0	3	1	5	1	-1	1	1
8	0	10	2	1	4	9	1	-1	-1	-1
8	1	11	3	2	5	10	-1	-1	-1	-1
8	2	6	4	3	0	11	1	-1	1	-1
8	3	7	5	4	1	6	1	-1	-1	-1
8	4	8	0	5	2	7	1	1	1	1
8	5	9	1	0	3	8	-1	1	-1	1
8	6	4	8	7	10	3	-1	1	-1	-1
8	7	5	9	8	11	4	1	1	1	1
8	8	0	10	9	6	5	-1	1	1	-1
8	9	1	11	10	7	0	1	-1	1	-1
8	10	2	6	11	8	1	-1	1	1	1
8	11	3	7	6	9	2	-1	-1	1	1
9	0	9	8	2	11	5	1	-1	1	1
9	1	10	9	3	6	0	1	-1	1	1
9	2	11	10	4	7	1	-1	1	1	1
9	3	6	11	5	8	2	1	1	-1	-1
9	4	7	6	0	9	3	-1	-1	-1	1
9	5	8	7	1	10	4	1	1	1	-1

9	6	3	2	8	5	11	1	1	-1	1
9	7	4	3	9	0	6	-1	1	1	-1
9	8	5	4	10	1	7	1	-1	-1	-1
9	9	0	5	11	2	8	1	1	-1	-1
9	10	1	0	6	3	9	-1	-1	-1	-1
9	11	2	1	7	4	10	-1	-1	-1	1
10	0	8	1	11	6	3	-1	1	1	1
10	1	9	2	6	7	4	1	-1	1	1
10	2	10	3	7	8	5	-1	1	1	1
10	3	11	4	8	9	0	1	1	-1	-1
10	4	6	5	9	10	1	1	-1	1	-1
10	5	7	0	10	11	2	1	1	1	-1
10	6	2	7	5	0	9	-1	-1	1	-1
10	7	3	8	0	1	10	1	1	-1	1
10	8	4	9	1	2	11	-1	1	1	1
10	9	5	10	2	3	6	1	1	1	-1
10	10	0	11	3	4	7	-1	-1	-1	-1
10	11	1	6	4	5	8	-1	1	1	-1
11	0	7	3	8	2	1	-1	-1	-1	1
11	1	8	4	9	3	2	1	1	-1	1
11	2	9	5	10	4	3	-1	1	-1	-1
11	3	10	0	11	5	4	-1	-1	-1	-1
11	4	11	1	6	0	5	1	1	-1	1
11	5	6	2	7	1	0	1	-1	1	-1
11	6	1	9	2	8	7	1	-1	-1	1
11	7	2	10	3	9	8	-1	-1	1	1
11	8	3	11	4	10	9	-1	1	-1	-1
11	9	4	6	5	11	10	-1	1	-1	-1
11	10	5	7	0	6	11	-1	-1	1	1
11	11	0	8	1	7	6	1	1	1	1

X'X

6072	4356	4356	4356	4356	4356	4356	0	0	0	0
	6072	4356	4356	4356	4356	4356	0	0	0	0
		6072	4356	4356	4356	4356	0	0	0	0
			6072	4356	4356	4356	0	0	0	-2
				6072	4356	4356	0	0	0	0
					6072	4356	0	0	0	0
						6072	0	0	0	0
							144	0	0	0
								144	0	0
									144	0
										144

R (Correlation matrix)

1	0.7174	0.7174	0.7174	0.7174	0.7174	0.7174	0	0	0	0
	1	0.7174	0.7174	0.7174	0.7174	0.7174	0	0	0	0
		1	0.7174	0.7174	0.7174	0.7174	0	0	0	0
			1	0.7174	0.7174	0.7174	0	0	0	-0.0021
				1	0.7174	0.7174	0	0	0	0
					1	0.7174	0	0	0	0
						1	0	0	0	0
							1	0	0	0
								1	0	0
									1	0
										1

inverse(R)

3.0599	-0.4786	-0.4786	-0.4786	-0.4786	-0.4786	-0.4786	0	0	0	-0.001
	3.0599	-0.4786	-0.4786	-0.4786	-0.4786	-0.4786	0	0	0	-0.001
		3.0599	-0.4786	-0.4786	-0.4786	-0.4786	0	0	0	-0.001
			3.0599	-0.4786	-0.4786	-0.4786	0	0	-0	0.0065
				3.0599	-0.4786	-0.4786	0	0	0	-0.001
					3.0599	-0.4786	0	0	0	-0.001
						3.0599	0	0	0	-0.001
							1	0	0	-0
								1	0	-0
									1	-0
										1

Table 1A. Output from NOA module using the array from Mandeli (1978) with an OLS(12, 5) set plus six F(12, 6) squares with one additional factor.

```
NOA 2.0: Construct orthogonal and near-orthogonal arrays with mixed levels
(C) 2001 Design Computing (URL: http://designcomputing.hypermart.net/gendex)

Note: design for m=14 and n=144.

try #               34
seed                -640488276
# of iterations     6
ave                 4451375.7802
max of s(i,j)       4356
freq of max         21
D-eff               0.4966
A-eff               0.4028

Factor levels:
```

0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1	1	1	1	1	1
0	2	2	2	2	2	2	1	1	1	1	1	1	-1
0	3	3	3	3	3	3	1	1	1	1	1	1	-1
0	4	4	4	4	4	4	1	1	1	1	1	1	-1
0	5	5	5	5	5	5	1	1	1	1	1	1	1
0	6	6	6	6	6	6	0	0	0	0	0	0	-1
0	7	7	7	7	7	7	0	0	0	0	0	0	-1
0	8	8	8	8	8	8	0	0	0	0	0	0	1
0	9	9	9	9	9	9	0	0	0	0	0	0	-1
0	10	10	10	10	10	10	0	0	0	0	0	0	-1
0	11	11	11	11	11	11	0	0	0	0	0	0	-1
1	0	5	6	3	10	2	0	1	1	0	0	0	-1
1	1	0	7	4	11	3	0	1	1	0	0	0	1
1	2	1	8	5	6	4	0	1	1	0	0	0	1
1	3	2	9	0	7	5	0	1	1	0	0	0	1
1	4	3	10	1	8	0	0	1	1	0	0	0	1
1	5	4	11	2	9	1	0	1	1	0	0	0	1
1	6	11	0	9	4	8	1	0	0	1	1	1	1
1	7	6	1	10	5	9	1	0	0	1	1	1	1
1	8	7	2	11	0	10	1	0	0	1	1	1	-1
1	9	8	3	6	1	11	1	0	0	1	1	1	1
1	10	9	4	7	2	6	1	0	0	1	1	1	-1
1	11	10	5	8	3	7	1	0	0	1	1	1	1
2	0	4	10	6	5	7	1	1	0	1	0	0	1
2	1	5	11	7	0	8	1	1	0	1	0	0	-1
2	2	0	6	8	1	9	1	1	0	1	0	0	-1
2	3	1	7	9	2	10	1	1	0	1	0	0	1
2	4	2	8	10	3	11	1	1	0	1	0	0	-1
2	5	3	9	11	4	6	1	1	0	1	0	0	1
2	6	10	4	0	11	1	0	0	1	0	1	1	-1
2	7	11	5	1	6	2	0	0	1	0	1	1	1
2	8	6	0	2	7	3	0	0	1	0	1	1	-1
2	9	7	1	3	8	4	0	0	1	0	1	1	-1
2	10	8	2	4	9	5	0	0	1	0	1	1	1
2	11	9	3	5	10	0	0	0	1	0	1	1	-1
3	0	3	4	5	7	8	0	0	0	1	1	0	-1
3	1	4	5	0	8	9	0	0	0	1	1	0	-1
3	2	5	0	1	9	10	0	0	0	1	1	0	1
3	3	0	1	2	10	11	0	0	0	1	1	0	1
3	4	1	2	3	11	6	0	0	0	1	1	0	1
3	5	2	3	4	6	7	0	0	0	1	1	0	-1
3	6	9	10	11	1	2	1	1	1	0	0	1	1
3	7	10	11	6	2	3	1	1	1	0	0	1	-1
3	8	11	6	7	3	4	1	1	1	0	0	1	1
3	9	6	7	8	4	5	1	1	1	0	0	1	-1
3	10	7	8	9	5	0	1	1	1	0	0	1	-1
3	11	8	9	10	0	1	1	1	1	0	0	1	1
4	0	2	11	9	1	4	0	0	0	0	1	1	1

4	1	3	6	10	2	5	0	0	0	0	1	1	-1
4	2	4	7	11	3	0	0	0	0	0	1	1	1
4	3	5	8	6	4	1	0	0	0	0	1	1	-1
4	4	0	9	7	5	2	0	0	0	0	1	1	1
4	5	1	10	8	0	3	0	0	0	0	1	1	1
4	6	8	5	3	7	10	1	1	1	1	0	0	1
4	7	9	0	4	8	11	1	1	1	1	0	0	-1
4	8	10	1	5	9	6	1	1	1	1	0	0	1
4	9	11	2	0	10	7	1	1	1	1	0	0	1
4	10	6	3	1	11	8	1	1	1	1	0	0	-1
4	11	7	4	2	6	9	1	1	1	1	0	0	-1
5	0	1	5	7	9	11	1	0	1	0	0	1	1
5	1	2	0	8	10	6	1	0	1	0	0	1	-1
5	2	3	1	9	11	7	1	0	1	0	0	1	-1
5	3	4	2	10	6	8	1	0	1	0	0	1	1
5	4	5	3	11	7	9	1	0	1	0	0	1	1
5	5	0	4	6	8	10	1	0	1	0	0	1	1
5	6	7	11	1	3	5	0	1	0	1	1	0	-1
5	7	8	6	2	4	0	0	1	0	1	1	0	1
5	8	9	7	3	5	1	0	1	0	1	1	0	1
5	9	10	8	4	0	2	0	1	0	1	1	0	1
5	10	11	9	5	1	3	0	1	0	1	1	0	1
5	11	6	10	0	2	4	0	1	0	1	1	0	-1
6	0	6	9	4	3	10	1	0	1	0	1	0	1
6	1	7	10	5	4	11	1	0	1	0	1	0	-1
6	2	8	11	0	5	6	1	0	1	0	1	0	-1
6	3	9	6	1	0	7	1	0	1	0	1	0	1
6	4	10	7	2	1	8	1	0	1	0	1	0	1
6	5	11	8	3	2	9	1	0	1	0	1	0	-1
6	6	0	3	10	9	4	0	1	0	1	0	1	-1
6	7	1	4	11	10	5	0	1	0	1	0	1	1
6	8	2	5	6	11	0	0	1	0	1	0	1	-1
6	9	3	0	7	6	1	0	1	0	1	0	1	1
6	10	4	1	8	7	2	0	1	0	1	0	1	-1
6	11	5	2	9	8	3	0	1	0	1	0	1	1
7	0	11	7	10	8	6	0	1	1	1	1	1	-1
7	1	6	8	11	9	7	0	1	1	1	1	1	-1
7	2	7	9	6	10	8	0	1	1	1	1	1	1
7	3	8	10	7	11	9	0	1	1	1	1	1	1
7	4	9	11	8	6	10	0	1	1	1	1	1	1
7	5	10	6	9	7	11	0	1	1	1	1	1	-1
7	6	5	1	4	2	0	1	0	0	0	0	0	1
7	7	0	2	5	3	1	1	0	0	0	0	0	-1
7	8	1	3	0	4	2	1	0	0	0	0	0	1
7	9	2	4	1	5	3	1	0	0	0	0	0	-1
7	10	3	5	2	0	4	1	0	0	0	0	0	-1
7	11	4	0	3	1	5	1	0	0	0	0	0	-1
8	0	10	2	1	4	9	0	1	0	0	0	1	-1
8	1	11	3	2	5	10	0	1	0	0	0	1	1
8	2	6	4	3	0	11	0	1	0	0	0	1	1
8	3	7	5	4	1	6	0	1	0	0	0	1	1
8	4	8	0	5	2	7	0	1	0	0	0	1	-1
8	5	9	1	0	3	8	0	1	0	0	0	1	-1
8	6	4	8	7	10	3	1	0	1	1	1	0	-1
8	7	5	9	8	11	4	1	0	1	1	1	0	-1
8	8	0	10	9	6	5	1	0	1	1	1	0	1
8	9	1	11	10	7	0	1	0	1	1	1	0	-1
8	10	2	6	11	8	1	1	0	1	1	1	0	-1
8	11	3	7	6	9	2	1	0	1	1	1	0	1
9	0	9	8	2	11	5	1	0	0	1	0	1	-1
9	1	10	9	3	6	0	1	0	0	1	0	1	-1
9	2	11	10	4	7	1	1	0	0	1	0	1	-1
9	3	6	11	5	8	2	1	0	0	1	0	1	1
9	4	7	6	0	9	3	1	0	0	1	0	1	1
9	5	8	7	1	10	4	1	0	0	1	0	1	1
9	6	3	2	8	5	11	0	1	1	0	1	0	1
9	7	4	3	9	0	6	0	1	1	0	1	0	-1
9	8	5	4	10	1	7	0	1	1	0	1	0	-1
9	9	0	5	11	2	8	0	1	1	0	1	0	-1
9	10	1	0	6	3	9	0	1	1	0	1	0	1
9	11	2	1	7	4	10	0	1	1	0	1	0	1

10	0	8	1	11	6	3	1	1	0	0	1	0	-1
10	1	9	2	6	7	4	1	1	0	0	1	0	-1
10	2	10	3	7	8	5	1	1	0	0	1	0	1
10	3	11	4	8	9	0	1	1	0	0	1	0	1
10	4	6	5	9	10	1	1	1	0	0	1	0	1
10	5	7	0	10	11	2	1	1	0	0	1	0	1
10	6	2	7	5	0	9	0	0	1	1	0	1	-1
10	7	3	8	0	1	10	0	0	1	1	0	1	-1
10	8	4	9	1	2	11	0	0	1	1	0	1	-1
10	9	5	10	2	3	6	0	0	1	1	0	1	1
10	10	0	11	3	4	7	0	0	1	1	0	1	1
10	11	1	6	4	5	8	0	0	1	1	0	1	1
11	0	7	3	8	2	1	0	0	1	1	0	0	-1
11	1	8	4	9	3	2	0	0	1	1	0	0	-1
11	2	9	5	10	4	3	0	0	1	1	0	0	1
11	3	10	0	11	5	4	0	0	1	1	0	0	-1
11	4	11	1	6	0	5	0	0	1	1	0	0	1
11	5	6	2	7	1	0	0	0	1	1	0	0	1
11	6	1	9	2	8	7	1	1	0	0	1	1	-1
11	7	2	10	3	9	8	1	1	0	0	1	1	-1
11	8	3	11	4	10	9	1	1	0	0	1	1	-1
11	9	4	6	5	11	10	1	1	0	0	1	1	1
11	10	5	7	0	6	11	1	1	0	0	1	1	1
11	11	0	8	1	7	6	1	1	0	0	1	1	-1

X'X

6072	4356	4356	4356	4356	4356	4356	396	396	396	396	396	396	-2
	6072	4356	4356	4356	4356	4356	396	396	396	396	396	396	0
		6072	4356	4356	4356	4356	396	396	396	396	396	396	2
			6072	4356	4356	4356	396	396	396	396	396	396	0
				6072	4356	4356	396	396	396	396	396	396	0
					6072	4356	396	396	396	396	396	396	0
						6072	396	396	396	396	396	396	0
							72	36	36	36	36	36	-2
								72	36	36	36	36	4
									72	36	36	36	0
										72	36	36	0
											72	36	0
												72	0
													144

R (Correlation matrix)

1	0.7174	0.7174	0.7174	0.7174	0.7174	0.7174	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	-0.0021
	1	0.7174	0.7174	0.7174	0.7174	0.7174	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	0
		1	0.7174	0.7174	0.7174	0.7174	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	0.0021
			1	0.7174	0.7174	0.7174	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	0
				1	0.7174	0.7174	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	0
					1	0.7174	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	0
						1	0.5989	0.5989	0.5989	0.5989	0.5989	0.5989	0
							1	0.5	0.5	0.5	0.5	0.5	-0.0196
								1	0.5	0.5	0.5	0.5	0.0393
									1	0.5	0.5	0.5	0
										1	0.5	0.5	0
											1	0.5	0
												1	0
													1

Note: the first 13 columns of the design are protected columns.
Note: NOA used 5.49 seconds.

Table 2. SAS/GLM code, data, and 19 F-squares.

```
data   fsquare;
input number rep treat A B AB C D CD AC AD ACD BC BD BD2 BCD BCD2
      ABC ABD ABD2 ABCD ABCD2 ;
datalines;
92   1  1  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
66   1  2  0  0  0  0  1  1  0  1  1  0  1  2  1  2  0  1  2  1  2
19   1  3  0  0  0  0  2  2  0  2  2  0  2  1  2  1  0  2  1  2  1
29   1  4  0  0  0  1  0  1  1  0  1  1  0  0  1  1  1  0  0  1  1
16   1  5  0  0  0  1  1  2  1  1  2  1  1  2  2  0  1  1  2  2  0
25   1  6  0  0  0  1  2  0  1  2  0  1  2  1  0  2  1  2  1  0  2
60   2  1  0  1  1  0  0  0  0  0  0  1  1  1  1  1  1  1  1  1  1
46   2  2  0  1  1  0  1  1  0  1  1  1  2  0  2  0  1  2  0  2  0
35   2  3  0  1  1  0  2  2  0  2  2  1  0  2  0  2  1  0  2  0  2
10   2  4  0  1  1  1  0  1  1  0  1  2  1  1  2  2  2  1  1  2  2
11   2  5  0  1  1  1  1  2  1  1  2  2  2  0  0  1  2  2  0  0  1
 5   2  6  0  1  1  1  2  0  1  2  0  2  0  2  1  0  2  0  2  1  0
46   3  1  0  2  2  0  0  0  0  0  0  2  2  2  2  2  2  2  2  2  2
81   3  2  0  2  2  0  1  1  0  1  1  2  0  1  0  1  2  0  1  0  1
17   3  3  0  2  2  0  2  2  0  2  2  2  1  0  1  0  2  1  0  1  0
22   3  4  0  2  2  1  0  1  1  0  1  0  2  2  0  0  0  2  2  0  0
16   3  5  0  2  2  1  1  2  1  1  2  0  0  1  1  2  0  0  1  1  2
 9   3  6  0  2  2  1  2  0  1  2  0  0  1  0  2  1  0  1  0  2  1
120  4  1  1  0  1  0  0  0  1  1  1  0  0  0  0  0  1  1  1  1  1
59   4  2  1  0  1  0  1  1  1  2  2  0  1  2  1  2  1  2  0  2  0
43   4  3  1  0  1  0  2  2  1  0  0  0  2  1  2  1  1  0  2  0  2
15   4  4  1  0  1  1  0  1  0  1  2  1  0  0  1  1  2  1  1  2  2
10   4  5  1  0  1  1  1  2  0  2  0  1  1  2  2  0  2  2  0  0  1
 2   4  6  1  0  1  1  2  0  0  0  1  1  2  1  0  2  2  0  2  1  0
49   5  1  1  1  2  0  0  0  1  1  1  1  1  1  1  1  2  2  2  2  2
64   5  2  1  1  2  0  1  1  1  2  2  1  2  0  2  0  2  0  1  0  1
25   5  3  1  1  2  0  2  2  1  0  0  1  0  2  0  2  2  1  0  1  0
24   5  4  1  1  2  1  0  1  0  1  2  2  1  1  2  2  0  2  2  0  0
 8   5  5  1  1  2  1  1  2  0  2  0  2  2  0  0  1  0  0  1  1  2
 7   5  6  1  1  2  1  2  0  0  0  1  2  0  2  1  0  0  1  0  2  1
134  6  1  1  2  0  0  0  0  1  1  1  2  2  2  2  2  0  0  0  0  0
60   6  2  1  2  0  0  1  1  1  2  2  2  0  1  0  1  0  1  2  1  2
52   6  3  1  2  0  0  2  2  1  0  0  2  1  0  1  0  0  2  1  2  1
20   6  4  1  2  0  1  0  1  0  1  2  0  2  2  0  0  1  0  0  1  1
28   6  5  1  2  0  1  1  2  0  2  0  0  0  1  1  2  1  1  2  2  0
11   6  6  1  2  0  1  2  0  0  0  1  0  1  0  2  1  1  2  1  0  2
run ;
proc glm data = fsquare;
class rep treat;
model number = rep treat;
run;
proc glm data = fsquare;
  class A B C D ;
  model number = A B A*B C D C*D A*C A*D A*C*D B*C B*D B*C*D
  A*B*C A*B*D A*B*C*D ;
run ;
proc glm data = fsquare;
class A B C D AB CD AC AD ACD BD BD2 BCD BCD2 ABC ABD ABD2
  ABCD ABCD2 BC;
model number = A B AB C D CD AC AD ACD BC BD BD2  BCD BCD2
```

```
ABC ABD ABD2 ABCD ABCD2 ;
RUN; QUIT;
```

Table 3. ANOVAs for four-factor factorial and for the 19 F-squares given in Table 2.

Source of variation	D.F.	Sum of squares	F-square	D.F	Sum of squares
Total	36	85,312.00			
Mean	1	49,580.44			
Row(Rep)	5	2,375.22			
A	1	441.00	FA	1	441.00
B	2	1,283.56	FB	2	1,283.56
A×B	2	650.67	FAB	2	650.67
Column(Tr)	5	26,196.22			
C	1	17,777.78	FC	1	17,777.78
D	2	5,783.39	FD	2	5,783.39
C×D	2	2,635.06	FCD	2	2,635.06
Row×Column	25	7,160.11			
A×C	1	729.00	FAC	1	729.00
A×D	2	522.17	FAD	2	522.17
A×C×D	2	624.50	FACD	2	624.50
B×C	2	360.22	FBC	2	360.22
B×D	4	843.11	FBD	2	707.72
			FBD2	2	135.39
B×C×D	4	805.44	FBCD	2	338.39
			FBCD2	2	467.06
A×B×C	2	612.67	FABC	2	612.67
A×B×D	4	763.67	FABD	2	487.50
			FABD2	2	276.17
A×B×C×D	4	1,899.33	FABCD	2	1,356.17
			FABCD2	2	543.17

Table 4. SAS/GLM output for combining a set of combinatorially orthogonal squares with a set of sum-of-squares orthogonal squares of order six.

Class Level Information

Class	Levels	Values
row	6	1 2 3 4 5 6
col	6	1 2 3 4 5 6
F1	3	0 1 2
F2	3	0 1 2
F3	3	0 1 2
F4	3	0 1 2
treat	6	0 1 2 3 4 5
F5	3	0 1 2
F6	3	0 1 2
F7	3	0 1 2

Number of observations 36

Dependent Variable: Y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	29	34569.22222	1192.04215	6.15	0.0153
Error	6	1162.33333	193.72222		
Corrected Total	35	35731.55556			

R-Square	Coeff Var	Root MSE	Y Mean
0.967470	37.50471	13.91841	37.11111

Source	DF	Type I SS	Mean Square	F Value	Pr > F
row	5	2375.22222	475.04444	2.45	0.1527
col	5	26196.22222	5239.24444	27.05	0.0005
F1	2	2.88889	1.44444	0.01	0.9926
F2	2	141.72222	70.86111	0.37	0.7081
F3	2	759.05556	379.52778	1.96	0.2214
F4	2	122.05556	61.02778	0.32	0.7411
treat	5	3468.88889	693.77778	3.58	0.0760
F5	2	8.22222	4.11111	0.02	0.9791
F6	2	204.22222	102.11111	0.53	0.6153
F7	2	1290.72222	645.36111	3.33	0.1064

Source	DF	Type III SS	Mean Square	F Value	Pr > F
row	5	2375.22222	475.04444	2.45	0.1527
col	5	26196.22222	5239.24444	27.05	0.0005
F1	2	2.88889	1.44444	0.01	0.9926
F2	2	141.72222	70.86111	0.37	0.7081
F3	2	759.05556	379.52778	1.96	0.2214
F4	2	122.05556	61.02778	0.32	0.7411
treat	5	3468.88889	693.77778	3.58	0.0760
F5	2	8.22222	4.11111	0.02	0.9791
F6	2	204.22222	102.11111	0.53	0.6153
F7	2	1290.72222	645.36111	3.33	0.1064

Class Level Information

Class	Levels	Values
row	6	1 2 3 4 5 6
col	6	1 2 3 4 5 6
F1	3	0 1 2
F2	3	0 1 2
F3	3	0 1 2
F4	3	0 1 2
treat	6	0 1 2 3 4 5
F5	3	0 1 2
F6	3	0 1 2
F7	3	0 1 2
A	2	0 1
B	3	0 1 2
C	2	0 1
D	3	0 1 2
E	2	0 1
F	3	0 1 2
AF1	3	0 1 2
AF2	3	0 1 2
CF3	3	0 1 2

Dependent Variable: Y

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	35	35731.55556	1020.90159	.	.
Error	0	0.00000	.		
Corrected Total	35	35731.55556			

R-Square	Coeff Var	Root MSE	Y Mean
1.000000	.	.	37.11111

Source	DF	Type I SS	Mean Square	F Value	Pr > F
row	5	2375.22222	475.04444	.	.
col	5	26196.22222	5239.24444	.	.
F1	2	2.88889	1.44444	.	.
F2	2	141.72222	70.86111	.	.
F3	2	759.05556	379.52778	.	.
F4	2	122.05556	61.02778	.	.
treat	5	3468.88889	693.77778	.	.
F5	2	8.22222	4.11111	.	.
F6	2	204.22222	102.11111	.	.
F7	2	1290.72222	645.36111	.	.
AF1	2	243.74444	121.87222	.	.
AF2	2	93.20958	46.60479	.	.
CF3	2	825.37931	412.68966	.	.

Source	DF	Type III SS	Mean Square	F Value	Pr > F
row	5	1358.96218	271.79244	.	.
col	5	11246.07052	2249.21410	.	.
F1	2	833.79124	416.89562	.	.
F2	2	768.00156	384.00078	.	.
F3	2	1144.89112	572.44556	.	.
F4	2	823.57255	411.78627	.	.
treat	5	898.96885	179.79377	.	.
F5	2	598.52059	299.26030	.	.
F6	2	693.81658	346.90829	.	.
F7	2	1755.80322	877.90161	.	.
AF1	2	637.86296	318.93148	.	.
AF2	2	775.98004	387.99002	.	.
CF3	2	825.37931	412.68966		